Yang–Baxter maps and multi-field integrable lattice equations

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Abstract

A variety of Yang–Baxter maps are obtained from integrable multi-field equations on quad-graphs. A systematic framework for investigating this connection relies on the symmetry groups of the equations. The method is applied to lattice equations introduced by Adler and Yamilov, which are related to the nonlinear superposition formulae for the Bäcklund transformations of the nonlinear Schrödinger system and specific ferromagnetic models.

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1. Introduction

The study of set-theoretic solutions of the quantum Yang–Baxter (YB) equation was originally suggested by Drinfeld in [1] and has attracted the interest of many researchers. Certain examples of such solutions had already appeared in the relevant literature by Sklyanin [2]. Set-theoretic solutions of the quantum YB equation, termed as classical solutions of the quantum YB equation, were investigated by Weinstein and Xu [3] from a geometric point of view using the theory of Poisson Lie groups and the associated symplectic groupoids. The algebraic theory of the subject was developed by Etingof, Schedler and Soloviev [4]. More recently, the dynamical theory of the set-theoretic solutions of the quantum YB equation was developed by Veselov in [5] (see also the review article [6]), where the notion of the associated transfer maps was introduced as a dynamical analogue of the transfer matrices of the theory of solvable models in statistical mechanics. Veselov suggested the short-term YB maps when referring to the set-theoretic (or classical) solutions of the YB equation, which we adopt in the present work.

Let X be a set. A map $R : X \times X \to X \times X$ is called YB map, if it satisfies the relation

$$R^{(2,3)} R^{(1,3)} R^{(1,2)} = R^{(1,2)} R^{(1,3)} R^{(2,3)}, \tag{1}$$

regarded as an equality of composite maps from $X \times X \times X$ into itself. Following the usual notation, $R^{(1,3)}$ is meant as the identity in the second factor of $X \times X \times X$ and as R in the first and third factors, and analogously for $R^{(1,2)}$, $R^{(2,3)}$ (see, e.g., [6]). In various interesting

examples of YB maps, such as maps arising from geometric crystals [7], the set X has the structure of an algebraic variety and R is a birational isomorphism. The case of $\mathbb{CP}^1 \times \mathbb{CP}^1$ has been recently discussed by Adler, Bobenko and Suris in [8], in relation with the classification of the so-called *quadrirational maps*. Remarkably, the resulting list of YB maps corresponds exactly to the five different ways of intersection of two conics. A classification for integrable equations on quad-graphs was also performed by Adler, Bobenko and Suris [9], where the connection between the YB relation for maps and the *consistency* property for discrete equation on quad-graphs was addressed (see concluding remarks of [9]).

More recently, the connection between YB maps and integrable equations on quad-graphs has been investigated in [10]. It was shown that a systematic framework for studying such a connection is based on the symmetry groups of the equations. The main observation is that the variables of certain YB maps can be chosen as invariants of the symmetry group of the corresponding lattice equation. Thus, it was demonstrated that YB maps can be obtained from scalar integrable lattice equations admitting a one-parameter group of symmetry transformations. Moreover, this idea was extended to scalar integrable lattice equations admitting a multi-parameter symmetry group by considering the extension of the equation on a multi-dimensional lattice as well as to multi-field integrable lattice equations.

In the present work, we continue the study on the connection between YB maps and multi-field integrable lattice equations. In section 2, we present the explicit auto-transformations of integrable chains introduced by Adler and Yamilov in [11] and the interpretation of these transformations as discrete equations on quad-graphs. Integrability aspects of such equations are discussed in section 3. There we stress the link between the three-dimensional consistency property of lattice equations on quad-graphs and the braid-type relation of the considered auto-transformations. Section 4 deals with symmetry methods applied to lattice equations. The existence of a symmetry group of transformations allows one to construct YB maps from integrable lattice equations by considering invariants of the transformation group as YB variables. The procedure is applied to the list of lattice equations of section 2 and the results are presented in section 5. In section 6, we derive YB maps from n-vector extensions of the nonlinear Schrödinger system. We conclude in section 7 by making comments on various perspectives of the subject.

2. Explicit auto-transformations of integrable chains as two-field quad-graph equations

Adler and Yamilov introduced explicit auto-transformations in [11] for integrable chains associated with the systems of evolution partial differential equations of the form

$$u_t = u_{xx} + f(u, v, u_x), \qquad v_t = -v_{xx} + g(u, v, v_x).$$

For example, the nonlinear Schrödinger system

$$u_t = u_{xx} + 2u^2v, \qquad v_t = -v_{xx} - 2uv^2$$

is related to the chain

$$u_{jx} = -u_{j+1} - \alpha_j u_j - u_j^2 v_{j+1}, \qquad v_{jx} = v_{j-1} + \alpha_{j-1} v_j + u_{j-1} v_j^2.$$

For given $k \in \mathbb{Z}$, this chain admits (up to the exchange of two consecutive α's) an explicit auto-transformation $\mathbb{B}_k : (u_j, v_j) \to (\widetilde{u}_j, \widetilde{v}_j)$, which preserves the values of the fields (u_j, v_j), $j \in \mathbb{Z}\backslash\{k\}$ and the parameters α_j, $j \in \mathbb{Z}\backslash\{k-1, k\}$ and changes the remaining terms of the chain by

$$\widetilde{u}_k = u_k + \frac{\alpha_{k-1} - \alpha_k}{1 - u_{k-1}v_{k+1}} u_{k-1}, \qquad \widetilde{v}_k = v_k - \frac{\alpha_{k-1} - \alpha_k}{1 - u_{k-1}v_{k+1}} v_{k+1}, \qquad (2a)$$
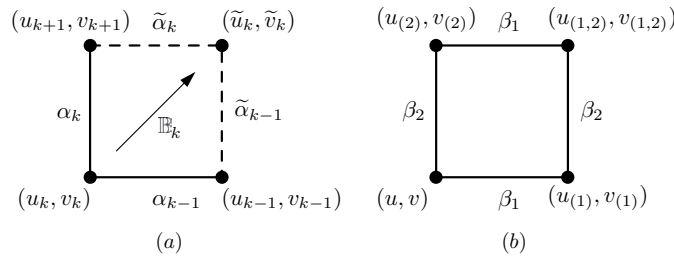
Figure 1. (a) An elementary flip representing the transformation $\mathbb{B}_k$. (b) The corresponding notation of the fields and the parameters.

$$\widetilde{\alpha}_k = \alpha_{k-1}, \qquad \tilde{\alpha}_{k-1} = \alpha_k. \tag{2b}$$

An interpretation of the above transformations can be given in terms of deformations of simple paths on the multi-dimensional orthogonal grid. This interpretation follows naturally if one starts from the two-dimensional grid situation studied in [12] and generalizes to higher dimension inspired by the multi-dimensional consistency [9, 13] and the Cauchy problem on quad-graphs [14].

We consider the multi-dimensional orthogonal grid with set of vertices $\mathbb{Z}^m$, edges connecting neighbouring vertices $(n_1, n_2, \ldots, n_p, \ldots, n_m)$, $(n_1, n_2, \ldots, n_p + 1, \ldots, n_m)$ and a set of m different complex parameters $\beta_1, \beta_2, \ldots, \beta_m$. For each $(n_1, n_2, \ldots, n_p, \ldots, n_m) \in \mathbb{Z}^m$ the edge between the vertices $(n_1, n_2, \ldots, n_p, \ldots, n_m)$ and $(n_1, n_2, \ldots, n_p + 1, \ldots, n_m)$ is labelled by the value β_p, so parallel edges are labelled with the same β-value. A plane perpendicular to an edge at its midpoint is called a characteristic plane. We consider now a simple path on the grid defined by a sequence (κ_j) of vertices with the property that each characteristic plane cuts one and only one edge of the path. This is an instance of a Cauchy path for quad-graph equations as was analysed in [14]. At the vertex κ_j we put the value of the two-field (u_j, v_j) and we assume that the value of the parameter α_j of the chain coincides with the β-value of the edge (κ_j, κ_{j+1}). Then, in the generic case when $\alpha_k \neq \alpha_{k-1}$, the transformation $\mathbb{B}_k$ can be thought as a local deformation replacing the $(u_{k-1}, v_{k-1}) \xrightarrow{\alpha_{k-1}} (u_k, v_k) \xrightarrow{\alpha_k} (u_{k+1}, v_{k+1})$ angular segment of the path by $(u_{k-1}, v_{k-1}) \xrightarrow{\alpha_k} (\tilde{u}_k, \tilde{v}_k) \xrightarrow{\alpha_{k-1}} (u_{k+1}, v_{k+1})$ as in figure 1(a), i.e. flipping one angle of the square face to the opposite. Note that if $\alpha_k = \alpha_{k-1}$, the transformation $\mathbb{B}_k$ reduces to the identity which is compatible with the fact that the corresponding two consecutive edges lie on the same line and there is no angle to flip.

Having this picture in mind we see that the basic building block is a square *plaquette* on a two-dimensional plane (e.g. in the 1 and 2 directions) of the grid with equations (2b) rewritten in the form

$$u_{(1,2)} = u + \frac{\beta_1 - \beta_2}{1 - u_{(1)}v_{(2)}}u_{(1)}, \qquad v_{(1,2)} = v - \frac{\beta_1 - \beta_2}{1 - u_{(1)}v_{(2)}}v_{(2)}, \tag{3}$$

where the subscript i inside a parenthesis indicates the displacement of the fields (u, v) in the i direction by a unit step on the orthogonal grid (see figure 1(b)). Following the above notation we list below the equations from [11] which are relevant in the present work. We denote them by $\mathcal{E}_a, a = 1, 2, \ldots, 5$,

$$\mathcal{E}_a: \quad u_{(1,2)} = E_a(u, u_{(1)}, v_{(2)}, \beta_1, \beta_2), \qquad v_{(1,2)} = E_a(v, v_{(2)}, u_{(1)}, \beta_2, \beta_1), \tag{4}$$

where the corresponding functions on the right-hand sides, evaluated at $(x, y, z, \beta_1, \beta_2)$, are given in table 1.
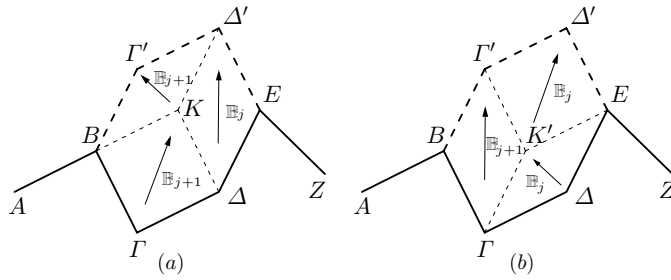
Figure 2. The braid relation as the consistency property for the deformations of a path on the multi-dimensional grid.

Table 1. The list of functions $E_a(x, y, z, \beta_1, \beta_2)$. The additional parameters γ_i, appearing in function E_5, depend on the parameters β_i through the relations $\gamma_i{}^2 - \beta_i{}^2 = \delta$, where δ can be fixed to be 0 or 1. In function E_4, ϵ is a free parameter.

$$E_1 = x + (\beta_1 - \beta_2)\frac{y}{1-yz} \qquad E_4 = x + \left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right)\frac{\beta_1\beta_2(x+z)(y-x)-\epsilon}{\beta_2(x+z)+\beta_1(y-x)}$$

$$E_2 = x + (\beta_2 - \beta_1)\frac{y-x}{\beta_2+yz}$$

$$E_3 = x + (\beta_1 - \beta_2)\frac{x+z}{y+z-\beta_1} \qquad E_5 = \frac{(\beta_1-\beta_2)yz+x(\gamma_2 y+\gamma_1 z)}{(\beta_1-\beta_2)x+\gamma_1 y+\gamma_2 z}$$

Remark 2.1. System $\mathcal{E}_1$ corresponds to equations (3). System $\mathcal{E}_2$ is related to the chain of the derivative nonlinear Schrödinger system [15] deformed by its first symmetry. System $\mathcal{E}_3$ is related to the chain associated with the Kaup system [16]. The remaining lattice equations $\mathcal{E}_4$ and $\mathcal{E}_5$ are related to chains corresponding to the Landau–Lifshits system

$$u_t = u_{xx} - \frac{2}{u+v}\left(u_x^2 + P(u)\right) + \frac{1}{2}P'(u), \qquad v_t = -v_{xx} + \frac{2}{u+v}\left(v_x^2 + P(-v)\right) + \frac{1}{2}P'(-v)$$

for the specific cases where $P(u) \equiv \epsilon$ (system $\mathcal{E}_4$) and $P(u) \equiv \delta u^2$ (system $\mathcal{E}_5$).

3. Braid relation and the three-dimensional consistency property

Key properties of the transformations $\mathbb{B}_j$ are the relations $\mathbb{B}_j^2 = 1$, $(\mathbb{B}_j\mathbb{B}_{j+1})^3 = 1$ and $\mathbb{B}_j\mathbb{B}_i = \mathbb{B}_i\mathbb{B}_j$, for $|i - j| > 1$. The first one means that each transformation $\mathbb{B}_j$ is an involution. The second one, in view of the first, yields the following braid-type relation:

$$\mathbb{B}_{j+1}\mathbb{B}_j\mathbb{B}_{j+1} = \mathbb{B}_j\mathbb{B}_{j+1}\mathbb{B}_j. \tag{5}$$

Let us now explain the implication of the braid relation (5) in terms of deformations of paths on the grid. By applying the composite transformations appearing on each side of equation (5) to a path $AB\Gamma\Delta EZ$, as shown in figure 2, we get two deformations. Consider first the left-hand side of the braid relation: $\mathbb{B}_{j+1}$ flips the Γ corner to K, then $\mathbb{B}_j$ flips the Δ corner to Δ' and finally $\mathbb{B}_{j+1}$ flips the K corner to Γ'. For the right-hand side, $\mathbb{B}_j$ flips the Δ corner to K' then $\mathbb{B}_{j+1}$ flips the Γ corner to Γ' and finally $\mathbb{B}_j$ flips the K' corner to Δ'. Thus, the original path $AB\Gamma\Delta EZ$ is transformed to the same path $AB\Gamma'\Delta'EZ$ through two different deformations according to figure 2, namely,

$$(a) \qquad AB\Gamma\Delta EZ \xrightarrow{\mathbb{B}_{j+1}} ABK\Delta EZ \xrightarrow{\mathbb{B}_j} ABK\Delta'EZ \xrightarrow{\mathbb{B}_{j+1}} AB\Gamma'\Delta'EZ,$$

$$(b) \qquad AB\Gamma\Delta EZ \xrightarrow{\mathbb{B}_j} AB\Gamma K'EZ \xrightarrow{\mathbb{B}_{j+1}} AB\Gamma'K'EZ \xrightarrow{\mathbb{B}j} AB\Gamma'\Delta'EZ.$$

The braid relation (5) guarantees that the values at the vertices Γ', Δ' are uniquely determined in terms of the values at the vertices B, Γ, Δ, E, i.e. independent of the way we deform the path in order to arrive at these vertices. This is an instance of the three-dimensional consistency for quad-graph equations (see [9, 13, 17] as well as [10]). The consistency property can be checked for an initial-value configuration as in figure 2, with values assigned on the vertices B, Γ, Δ, E. This configuration is adapted to the cubic interpretation of the YB property [18], cf also [10].

4. Invariants of symmetry groups and Yang–Baxter variables

We start with the building block of equations on quad-graphs consisting of a system of algebraic relations of the form

$$\mathcal{B}^i(f, f_{(1)}, f_{(2)}, f_{(1,2)}; \beta_1, \beta_2) = 0, \qquad i = 1, \ldots, \ell, \tag{6}$$

where $f = (f^1, f^2, \ldots, f^\ell)$ and $f^i : \mathbb{Z}^2 \to \mathbb{C}$ (or $\mathbb{CP}^1$), $i = 1, 2, \ldots, \ell$. We use upper indices to denote the components of the fields, since we have reserved lower indices to indicate their shifted values. Consider a one-parameter group of transformations G acting on the domain of the dependent variables of a lattice equation (6), i.e.

$$G : f^i \mapsto \Phi^i(n_1, n_2, f; \varepsilon), \qquad \varepsilon \in \mathbb{C}, \quad i = 1, \ldots, \ell,$$

where $n_1, n_2 \in \mathbb{Z}$ denote the site variables of the lattice. Let $\mathrm{J}^{(k)}$ denote the lattice jet space with coordinates (f, f_J), where by f_J we mean the forward shifted values of f, indexed by all unordered (symmetric) multi-indices $J = (j^1, j^2, \ldots, j^k)$, $1 \leqslant j^r \leqslant 2$, of order $k = \#J$. The prolongation of the group action of G on $\mathrm{J}^{(k)}$ is

$$G^{(k)} : (f^i, f_J^i) \mapsto (\Phi^i(n_1, n_2, f; \varepsilon), \Phi_J^i(n_1, n_2, f; \varepsilon)), \qquad i = 1, \ldots, \ell, \tag{7}$$

where $\Phi_{(1)}^i(n_1, n_2, f; \varepsilon) = \Phi^i(n_1 + 1, n_2, f_{(1)}; \varepsilon)$, etc.

The infinitesimal generator of the group action of G on the space of the dependent variables is given by the vector field

$$\mathbf{v} = \sum_{i=1}^n Q^i(n_1, n_2, f)\partial_{f^i}, \qquad \text{where} \qquad Q^i(n_1, n_2, f) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Phi^i(n_1, n_2, f; \varepsilon)\bigg|_{\varepsilon=0}.$$

There is a one-to-one correspondence between connected groups of transformations and their associated infinitesimal generators, since the group action is reconstructed by the flow of the vector field $\mathbf{v}$ by exponentiation

$$\Phi^i(n_1, n_2, f; \varepsilon) = \exp(\varepsilon \mathbf{v}) f^i, \qquad i = 1, \ldots, n.$$

The infinitesimal generator of the action of $G^{(k)}$ on $\mathrm{J}^{(k)}$ is the associated kth order prolonged vector field

$$\mathbf{v}^{(k)} = \sum_{i=1}^n \sum_{\#J=j=0}^k Q_J^i(n_1, n_2, f)\partial_{f_J^i}.$$

The transformation G is a Lie-point symmetry of the lattice equations (6), if it transforms any solution of (6) to another solution of the same equations. Equivalently, G is a symmetry of equations (6), if the equations are not affected by the transformation (7). The infinitesimal criterion for a connected group of transformations G to be a symmetry of equations (6) is

$$\mathbf{v}^{(2)}(\mathcal{B}^i(f, f_{(1)}, f_{(2)}, f_{(1,2)}; \beta_1, \beta_2)) = 0, \tag{8}$$

where $i = 1, 2 \ldots, n$. Equations (8) should hold for all solutions of equations (6), and thus the latter equations and their consequences should be taken into account. Equations (8)

determine the most general infinitesimal Lie-point symmetry of the system (6). The resulting set of infinitesimal generators forms a Lie algebra $\mathfrak{g}$ from which the corresponding Lie-point symmetry group G can be found by exponentiating the given vector fields.

A *lattice invariant* under the action of G is a function $I : \mathrm{J}^{(k)} \to \mathbb{C}$ which satisfies $I(g^{(k)} \cdot (f, f_J)) = I(f, f_J)$ for all $g \in G$ and all $(f, f_j) \in \mathrm{J}^{(k)}$. For connected groups of transformations, a necessary and sufficient condition for a function I to be invariant under the action of G is the annihilation of I by all prolonged infinitesimal generators, i.e.

$$\mathbf{v}^{(k)}(I) = 0, \tag{9}$$

for all $\mathbf{v} \in \mathfrak{g}$.

Consider a system of lattice equations of the form (6) ($\ell = 2$) and suppose that it admits a two-parameter group of symmetry transformations G generated by two vector fields $\{\mathbf{v}_1, \mathbf{v}_2\}$. The invariants under the action of G on $\mathrm{J}^{(2)}$ are obtained from the general solution of the system of first-order partial differential equations (9), by using the method of characteristics. Using these invariant functions I^i, we assign now to the edges of an elementary quadrilateral the following YB variables:

$$x^i = I^i(f, f_{(1)}), \qquad y^i = I^i(f_{(2)}, f), \qquad p^i = I^i(f_{(2)}, f_{(1,2)}), \qquad q^i = I^i(f_{(1,2)}, f_{(1)}). \tag{10}$$

The invariants x, y, p, q, where $x = (x^1, x^2)$, etc, are not independent; there exist two relations among them

$$\mathcal{F}^i(x, y, u, v) = 0, \qquad i = 1, 2, \tag{11}$$

following from the fact that the space of G-orbits on $\mathrm{J}^{(2)}$ is six dimensional. Moreover, since G is a symmetry group of the system of equations (6), it can be written in terms of the variables (10), yielding two more relations among the invariants x, y, p, q, i.e.

$$\mathcal{D}^i(x, y, p, q; \beta_1, \beta_2) = 0, \qquad i = 1, 2. \tag{12}$$

Solving the system of equations (12), (11) for p, q in terms of x, y and assuming that the solution is unique, we obtain a two-parameter family of maps $R(\beta_1, \beta_2) : (x, y) \mapsto (p, q)$. The resulting map R satisfies automatically the parameter-dependent YB relation:

$$R^{(2,3)}(\beta_2, \beta_3) R^{(1,3)}(\beta_1, \beta_3) R^{(1,2)}(\beta_1, \beta_2) = R^{(1,2)}(\beta_1, \beta_2) R^{(1,3)}(\beta_1, \beta_3) R^{(2,3)}(\beta_2, \beta_3). \tag{13}$$

This fact follows immediately from the interpretation of the braid relation (5) and the initial-value problem discussed at the end of the previous section, in terms of YB variables (see figure 3). More precisely, the LHS and RHS of the braid relation (5) correspond to the composition of mappings (a) and (b), respectively as follows:

$$(a) \quad (x, y, z) \xrightarrow{R^{(1,2)}} (x', y', z) \xrightarrow{R^{(1,3)}} (x'', y', z') \xrightarrow{R^{(2,3)}} (x'', y'', z''),$$

$$(b) \quad (x, y, z) \xrightarrow{R^{(2,3)}} (x, y^*, z^*) \xrightarrow{R^{(1,3)}} (x^*, y^*, z^{**}) \xrightarrow{R^{(1,2)}} (x^{**}, y^{**}, z^{**}).$$

The preceding considerations can be casted in the form of the following:

Proposition 4.1. *If the system of discrete equations $\mathcal{B}^i = 0, i = 1, 2$, satisfies the three-dimensional consistency property and admits a two-parameter group of symmetry transformations, then the map $R(\beta_1, \beta_2)$ which relates the lattice invariants (10), satisfies the parameter-dependent YB relation.*

The above framework provides a method for obtaining a YB map from a three-dimensional consistent lattice system of the form (6) ($\ell = 2$). However, there are circumstances where invariance of a system of lattice equations under a one-parameter group of transformations
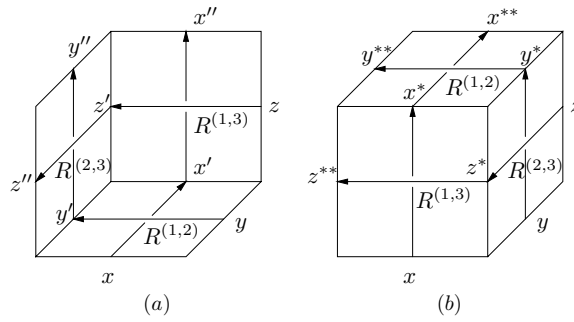
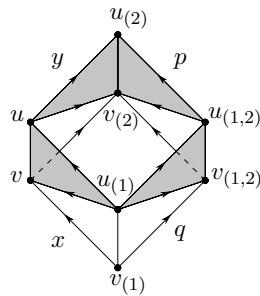**Figure 3.** Three-dimensional representation of the YB relation.

Figure 4. Two-field lattice equations: the values of the fields are assigned on the vertices of a cube, while the YB variables on the faces.

suffices to construct an associated YB map R. This is the case for all lattice equations of table 1, where we make the identification $f = (u, v)$. In order to visualize better this situation we are going to consider a cube with base the elementary square and place the second field on the top face as shown in figure 4.

Let us examine first the system $\mathcal{E}_1$ which, for convenience, we rewrite as

$$u_{(1,2)} = u + \frac{\beta_1 - \beta_2}{1 - u_{(1)} v_{(2)}} u_{(1)}, \qquad v_{(1,2)} = v - \frac{\beta_1 - \beta_2}{1 - u_{(1)} v_{(2)}} v_{(2)}. \tag{14}$$

Note that the values $u_{(2)}, v_{(1)}$ are not involved in the equations. Clearly, equations (14) are invariant under the one-parameter group of transformations

$$G : (u, v) \mapsto (e^\varepsilon u, e^{-\varepsilon} v), \tag{15}$$

with corresponding infinitesimal generator

$$\mathbf{v}_1 = u \partial_u - v \partial_v. \tag{16}$$

We consider now the space $\mathrm{J}^{(2)}\big(\overset{\vee}{v}_{(1)}\big)$, i.e. the space obtained from $\mathrm{J}^{(2)}$ by factoring out $v_{(1)}$. On $\mathrm{J}^{(2)}\big(\overset{\vee}{v}_{(1)}\big)$ we define the following lattice invariants along the orbits of $\mathbf{v}_1$:

$$(x^1, x^2) = (u/u_{(1)}, v u_{(1)}), \qquad (p^1, p^2) = (u_{(2)}/u_{(1,2)}, v_2 u_{(1,2)}), \tag{17a}$$

$$(y^1, y^2) = (u_{(2)}/u, u v_{(2)}), \qquad (q^1, q^2) = (u_{(1,2)}/u_{(1)}, u_{(1)} v_{(1,2)}). \tag{17b}$$

The space of G-orbits on $\mathrm{J}^{(2)}\big(\overset{\vee}{v}_{(1)}\big)$ is six dimensional and thus there exist exactly two functionally independent relations among the invariants (17), namely,

$$p^1 q^1 = x^1 y^1, \qquad p^1 p^2 = y^1 y^2. \tag{18}$$

Moreover, since G is a symmetry, equations (14) can be written in terms of the invariants (17), in the following form:

$$p^2 = y^2 P, \qquad q^2 = x^2 + y^2(1 - P), \qquad \text{where} \qquad P = 1 + \frac{\beta_1 - \beta_2}{x^1 - y^2}. \tag{19}$$

Solving the system of algebraic equations (18), (19) for (p, q) in terms of (x, y), we obtain the unique solution

$$(p^1, p^2) = (y^1 P^{-1}, y^2 P), \qquad (q^1, q^2) = (x^1 P, x^2 + y^2(1 - P)), \tag{20}$$

which defines the YB map $R : (x^i, y^i) \mapsto (p^i, q^i)$ (shaded triangles of figure 4).

On the other hand, one could equally well consider on the space $\mathrm{J}^{(2)}\big(\overset{\vee}{u}_{(2)}\big)$ the following invariants under the action of G (blank triangles of figure 4):

$$(x^3, x^2) = (v/v_{(1)}, v u_{(1)}), \qquad (p^3, p^2) = (v_{(2)}/v_{(1,2)}, v_{(2)} u_{(1,2)}), \tag{21a}$$

$$(y^3, y^2) = (v_{(2)}/v, u v_{(2)}), \qquad (q^3, q^2) = (v_{(1,2)}/v_1, u_{(1)} v_{(1,2)}). \tag{21b}$$

The above invariants are related by

$$p^3 q^3 = x^3 y^3, \qquad p^3 q^2 = y^3 x^2, \tag{22}$$

while equations (14) can be written in terms of them as follows:

$$p^2 = y^2 + x^2(1 - \widetilde{P}), \qquad q^2 = x^2 \widetilde{P}, \qquad \text{where} \qquad \widetilde{P} = 1 - \frac{(\beta_1 - \beta_2) y^3}{1 - x^2 y^3}. \tag{23}$$

Solving the above equations for (p^i, q^i), we obtain the YB map

$$(p^3, p^2) = (y^3 \widetilde{P}^{-1}, y^2 + x^2(1 - \widetilde{P})), \qquad (q^3, q^2) = (x^3 \widetilde{P}, x^2 \widetilde{P}). \tag{24}$$

The variables (17) (here called Yang–Baxter variables) were introduced in a different context in [11], where the reduction of the lattice equation (3) to the map (20) was also presented, together with an associated Lax pair. It seems that one of the motivations in [11] for this reduction was the illustration of the discrete Liouville theorem for integrable maps.

We close this section by presenting a Lax pair, different from the one given in [11], for the YB map (20). The YB map (20) admits the discrete zero curvature representation

$$W(y^1, y^2, \beta_2, \lambda) W(x^1, x^2, \beta_1, \lambda) = W(p^1, p^2, \beta_1, \lambda) W(q^1, q^2, \beta_2, \lambda),$$

where

$$W(\xi^1, \xi^2, \beta, \lambda) = \begin{pmatrix} \xi^1 & -\xi^1 \xi^2 \\ 1 & \lambda - \beta - \xi^2 \end{pmatrix}.$$

A hint for considering the matrix W of the above form is based on the YB map itself. Indeed, in view of equations (20) giving p^1, one notes that

$$p^1 = y^1 \frac{x^1 - y^2}{x^1 - y^2 + \beta_1 - \beta_2}$$

can be written as a linear fractional transformation induced from the linear transformation with matrix $W(y^1, y^2, \beta_2, \beta_1)$ on $(x^1, 1)$ (column vector), cf [18].

5. YB maps arising from equations $\mathcal{E}_\alpha$

In this section, we present the YB maps obtained from the lattice equations $\mathcal{E}_\alpha$ by considering invariants of their symmetry groups. For equations admitting a two-parameter symmetry group we consider as YB variables a complete set of joint invariants on $\mathrm{J}^{(2)}$. In the case of a one-parameter symmetry group, as we discussed in the previous section, one can consider two cases for the YB variables: (i) lattice invariants on the space $\mathrm{J}^{(2)}\big(v_{(1)}^{\vee}\big)$ (shaded triangles in figure 4) or (ii) lattice invariants on the space $\mathrm{J}^{(2)}\big(u_{(2)}^{\vee}\big)$ (blank triangles in figure 4). We present the YB maps obtained from case (i) only. For each system, the analysis of case (ii) yields identical map as in case (i), up to appropriate rearrangement of variables.

5.1. System $\mathcal{E}_2$

Equations $\mathcal{E}_2$ admit the symmetry generator (16), thus we take as YB variables those defined by equations (17), leading to the same functional relations (18). On the other hand, system $\mathcal{E}_2$ can be written in terms of the invariants (17) as

$$p^2 = y^2 P, \qquad q^2 = x^2 Q, \tag{25}$$

where

$$P = 1 + (\beta_2 - \beta_1)\frac{1 - x^1}{\beta_2 x^1 + y^2}, \qquad Q = 1 + (\beta_1 - \beta_2)\frac{y^2/x^2 - x^1}{\beta_1 x^1 + y^2}.$$

Solving the system of equations (18), (25) for (p^i, q^i) we obtain the YB map

$$(p^1, p^2) = (y^1 P^{-1}, y^2 P), \quad (q^1, q^2) = (x^1 P, x^2 Q).$$

5.2. System $\mathcal{E}_3$

Equations $\mathcal{E}_3$ admit the symmetry generator

$$\mathbf{v}_2 = \partial_u - \partial_v. \tag{26}$$

On the space $\mathrm{J}^{(2)}\big(v_{(1)}^{\vee}\big)$ we define as YB variables the following invariants:

$$(x^1, x^2) = (u - u_{(1)}, v + u_{(1)}), \qquad (p^1, p^2) = (u_{(2)} - u_{(1,2)}, v_{(2)} + u_{(1,2)}), \tag{27a}$$

$$(y^1, y^2) = (u_{(2)} - u, u + v_{(2)}), \qquad (q^1, q^2) = (u_{(1,2)} - u_{(1)}, u_{(1)} + v_{(1,2)}). \tag{27b}$$

They are related by

$$p^1 + q^1 = x^1 + y^1, \qquad p^1 + p^2 = y^1 + y^2 \tag{28}$$

and system $\mathcal{E}_3$ is written in terms of the invariants (27) as follows:

$$p^2 = y^2 P, \qquad q^2 = x^2 P^{-1}, \tag{29}$$

where

$$P = \frac{y^2 - x^1 - \beta_2}{y^2 - x^1 - \beta_1}.$$

Solving the system of equations (28), (29) for (p^i, q^i) we obtain the YB map

$$(p^1, p^2) = (y^1 + y^2(1 - P), y^2 P), \qquad (q^1, q^2) = (x^1 - y^2(1 - P), x^2 P^{-1}).$$

5.3. System $\mathcal{E}_4$

In the generic case $\epsilon \neq 0$, the YB variables are the invariants of the symmetry group generated by the vector field (26). System $\mathcal{E}_4$ is written in terms of the invariants (27) as follows:

$$p^2 = y^2 - P, \qquad q^2 = x^2 + Q, \tag{30}$$

where

$$P = \left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right)\frac{\beta_1\beta_2 y^2 x^1 + \epsilon}{\beta_2 y^2 - \beta_1 x^1}, \qquad Q = \left(\frac{1}{\beta_2} - \frac{1}{\beta_1}\right)\frac{\beta_1\beta_2 x^2(y^2 - x^1 - x^2) - \epsilon}{\beta_1 x^2 + \beta_2(y^2 - x^1 - x^2)}.$$

Inserting equations (30) into (28) and solving for (p^1, q^1) in terms of (x^i, y^i), we arrive at the YB map

$$(p^1, p^2) = (y^1 + P, y^2 - P), \qquad (q^1, q^2) = (x^1 - P, x^2 + Q).$$

We examine now the case $\epsilon = 0$. In this case, there exists, in addition, the scaling symmetry generated by the vector field

$$\mathbf{v}_3 = u\partial_u + v\partial_v. \tag{31}$$

On $\mathrm{J}^{(2)}\big(\overset{\vee}{v}_{(1)}\big)$ we define as YB variables the following invariants under $\mathbf{v}_3$

$$(x^1, x^2) = (u/u_{(1)}, v/u_{(1)}), \qquad (p^1, p^2) = (u_{(2)}/u_{(1,2)}, v_{(2)}/u_{(1,2)}), \tag{32a}$$

$$(y^1, y^2) = (u_{(2)}/u, v_{(2)}/u), \qquad (q^1, q^2) = (u_{(1,2)}/u_{(1)}, v_{(1,2)}/u_{(1)}), \tag{32b}$$

which are related by

$$p^1 q^1 = x^1 y^1, \qquad y^2 p^1 = y^1 p^2. \tag{33}$$

System $\mathcal{E}_4$ ($\epsilon = 0$) can be written in terms of the invariants (32), as follows:

$$p^1 = x^1 P, \qquad q^2 = x^2 Q, \tag{34}$$

where

$$P = 1 + \frac{(\beta_2 - \beta_1)(1 + y^2)(1 - x^1)}{\beta_2 x^1(1 + y^2) + \beta_1(1 - x^1)},$$

$$Q = 1 + \frac{1}{x^2}\frac{(\beta_1 - \beta_2)(1 + x^2)(x^1 y^2 - x^2)}{\beta_1(1 + x^2) + \beta_2(x^1 y^2 - x^2)}.$$

From relations (33), (34) we obtain the following YB map:

$$(p^1, p^2) = (y^1 P^{-1}, y^2 P^{-1}), \qquad (q^1, q^2) = (x^1 P, x^2 Q).$$

Finally, considering the group generated by $\{\mathbf{v}_2, \mathbf{v}_3\}$, given by (26), (31), we take on $\mathrm{J}^{(2)}$ the following joint invariants:

$$(x^1, x^2) = \left(\frac{u - u_{(1)}}{v + u_{(1)}}, \frac{v - v_{(1)}}{v + u_{(1)}}\right), \qquad (p^1, p^2) = \left(\frac{u_{(2)} - u_{(1,2)}}{v_{(2)} + u_{(1,2)}}, \frac{v_{(2)} - v_{(1,2)}}{v_{(2)} + u_{(1,2)}}\right), \tag{35a}$$

$$(y^1, y^2) = \left(\frac{u_{(2)} - u}{u + v_{(2)}}, \frac{v_{(2)} - v}{u + v_{(2)}}\right), \qquad (q^1, q^2) = \left(\frac{u_{(1,2)} - u_{(1)}}{u_{(1)} - v_{(1,2)}}, \frac{v_{(1,2)} - v_{(1)}}{u_{(1)} + v_{(1,2)}}\right). \tag{35b}$$

The above invariants are functionally related by

$$\frac{(1 + x^1)(1 + y^1)}{(1 + p^1)(1 + q^1)} = \frac{(1 - x^2)(1 - y^2)}{(1 - p^2)(1 - q^2)} = \frac{1 + x^1 y^2}{1 + p^2 q^1}. \tag{36}$$

In terms of the invariants (35), the system $\mathcal{E}_4$ ($\epsilon = 0$) can be written as follows:

$$q^1 = x^1 P, \qquad p^2 = y^2 P^{-1}, \tag{37}$$

where

$$P = \frac{\beta_1(1 - y^2) + \beta_2 y^2(1 + x^1)}{\beta_1 x^1(y^2 - 1) + \beta_2(1 + x^1)}.$$

Using relations (36), (37) we obtain the YB map

$$(p^1, p^2) = \left(\frac{(1 + x^1)(1 + y^1)}{1 + x^1 P} - 1, y^2 P^{-1} \right), \tag{38a}$$

$$(q^1, q^2) = \left(x^1 P, \frac{(1 - x^2)(1 - y^2)}{y^2 P^{-1} - 1} + 1 \right). \tag{38b}$$

5.4. System $\mathcal{E}_5$

We examine first the generic case $\delta = 1$, i.e. $\gamma_i^2 - \beta_i^2 = 1$. System $\mathcal{E}_5$ admits the scaling symmetry with generator $\mathbf{v}_3$, thus we use the invariants (32). In terms of these variables the system $\mathcal{E}_5$ is written as follows:

$$p^1 = x^1 P, \qquad q^2 = x^2 Q, \tag{39}$$

where

$$P = \frac{(\beta_1 - \beta_2)y^2 + \gamma_2 + \gamma_1 x^1 y^2}{(\beta_1 - \beta_2)x^1 + \gamma_1 + \gamma_2 x^1 y^2}, \qquad Q = \frac{(\beta_2 - \beta_1)\frac{x^1 y^2}{x^2} + \gamma_2 + \gamma_1 x^1 y^2}{(\beta_2 - \beta_1)x^2 + \gamma_1 + \gamma_2 x^1 y^2}.$$

The YB map obtained from relations (33), (39) is

$$(p^1, p^2) = (y^1 P^{-1}, y^2 P^{-1}), \qquad (q^1, q^2) = (x^1 P, x^2 Q).$$

When $\delta = 0$, the system $\mathcal{E}_5$ becomes system $\mathcal{E}_4$ with $\epsilon = 0$ directly (case $\gamma_i = -\beta_i$) or indirectly under the gauge transformation $(u, v) \mapsto (-1)^{n_1 + n_2}(u, v)$ (case $\gamma_i = \beta_i$).

6. Discrete vector Schrödinger system

In this section, we show how the preceding considerations can be generalized to vector extensions of integrable lattice equations yielding n-vector (two-field) YB maps. This is illustrated for the lattice n-vector Schrödinger system

$$\boldsymbol{u}_{(1,2)} = \boldsymbol{u} + \frac{\beta_1 - \beta_2}{1 - \boldsymbol{u}_{(1)} \cdot \boldsymbol{v}_{(2)}} \boldsymbol{u}_{(1)}, \qquad \boldsymbol{v}_{(1,2)} = \boldsymbol{v} + \frac{\beta_2 - \beta_1}{1 - \boldsymbol{u}_{(1)} \cdot \boldsymbol{v}_{(2)}} \boldsymbol{v}_{(2)}, \tag{40}$$

introduced by Adler [19], where $\boldsymbol{u} = (u^1, \ldots, u^n)$ etc and $\cdot$ stands for the usual Euclidean inner product. Equations (40) represent the superposition formula of the Bäcklund transformation for the n-vector nonlinear Schrödinger system found by Manakov [20].

System (40) remains invariant under the group of transformations generated by the vector fields

$$\mathbf{v}_i = u^i \partial_{u^i} - v^i \partial_{v^i},$$

$i = 1, \ldots, n$, where no summation on indices is assumed. In order to make the presentation more concise, we represent the vectors $\boldsymbol{u}, \boldsymbol{v}$ and their shifts in terms of diagonal matrices

denoted by the corresponding capital letter, e.g. $\boldsymbol{u} \to U = \mathrm{diag}(u^1, u^2, \ldots, u^n)$. On the lattice jet space $\mathrm{J}^{(2)}(\overset{\vee}{\boldsymbol{v}}_{(1)})$ we consider the following YB variables:

$$(X^1, X^2) = \left(UU_{(1)}{}^{-1}, VU_{(1)}\right), \qquad (P^1, P^2) = \left(U_{(2)}U_{(1,2)}{}^{-1}, V_{(2)}U_{(1,2)}\right),$$
$$(Y^1, Y^2) = \left(U_{(2)}U^{-1}, UV_{(2)}\right), \qquad (Q^1, Q^2) = \left(U_{(1,2)}U_{(1)}{}^{-1}, U_{(1)}V_{(1,2)}\right).$$

The above invariants are functionally related by

$$P^1Q^1 = X^1Y^1, \qquad P^1P^2 = Y^1Y^2 \tag{41}$$

and the lattice equations (40) are written in terms of them as follows:

$$P^2 = Y^2S, \qquad P^2 + Q^2 = X^2 + Y^2, \tag{42}$$

where

$$S = I + \frac{(\beta_1 - \beta_2)(X^1)^{-1}}{1 - \mathrm{tr}(Y^2(X^1)^{-1})}.$$

Solving equations (41), (42) for (P^i, Q^i) in terms of (X^i, Y^i) we obtain the YB map

$$(P^1, P^2) = (Y^1S^{-1}, Y^2S), \qquad (Q^1, Q^2) = (X^1S, X^2 + Y^2(I - S)). \tag{43}$$

It would be interesting to investigate the relation of the YB maps studied recently in [21–23] with the YB map (43) after imposing appropriate reality conditions on the latter, as it was done for the system $\mathcal{E}_1$ in [11].

7. Conclusions and perspectives

We have applied symmetry methods to multi-field integrable lattice equations in order to construct YB maps. The main idea is that the invariants of the symmetry groups of the lattice equations serve as variables of the associated YB maps.

According to [8], a birational map $R : (x, y) \mapsto (p(x, y), q(x, y))$ is called quadrirational if the maps $s \mapsto p(s, y)$ and $t \mapsto q(x, t)$ for generic values of y, x, are invertible. This property is equivalent to the nondegeneracy property which is often imposed additionally in the study of YB maps, cf [4]. We have to note that all YB maps we presented in this work do not satisfy the quadrirationality property. This can be seen immediately since (p^1, p^2) is independent of x^2. Two additional properties which characterize certain the YB maps are invertibility and unitarity. A YB map $R(\beta_1, \beta_2)$ with parameters (β_1, β_2) is called unitary if it satisfies the relation

$$R^{(2,1)}(\beta_2, \beta_1)R(\beta_1, \beta_2) = \mathrm{id},$$

where $R^{(2,1)} = PRP$ and P is the permutation map, i.e. $P : (x, y) \mapsto (y, x)$. The YB maps we obtained here are invertible and unitary.

Recently, Reshetikhin and Veselov [24] studied the Hamiltonian theory of the YB maps through Lie–Poisson structures. The Poisson structure corresponding to the auto-transformation (3) was presented in [11]. It would be interesting to explore the Poisson structure of the remaining auto-transformations and the associated YB maps using the theory of Poisson Lie groups as in [3, 24].

The protagonistic role of the YB relation in quantum integrability is well known. In view of recent developments (see [6] and references therein) it seems that the YB relation for maps acquires a prominent role in (classical) discrete integrability as well. Here, we saw that integrable lattice equations admitting a symmetry group can be reduced to YB maps. It

remains an open question how integrable lattice equations which do not admit an appropriate symmetry group can be casted in the form of a YB map.

By the way of studying the multi-field lattice equations introduced by Adler and Yamilov the question of classification of two-field lattice equations by imposing the three-dimensional consistency property is naturally raised. Before tackling this problem in full generality one can consider the restricted class of equations of type (4). It would be interesting to investigate whether the classification is exhausted essentially by the lattice equations in [11].

Acknowledgments

References

[1] Drinfeld V G 1992 On some unsolved problems in quantum group theory *Quantum groups (Leningrad, 1990) (Lecture Notes in Mathematics* vol 1510*)* ed P P Kulish (Berlin: Springer) pp 1–8
[2] Sklyanin E K 1988 Classical limits of $SU(2)$-invariant solutions of the Yang–Baxter equation *J. Sov. Math.* **40** 93–107
[3] Weinstein A and Xu P 1992 Classical solutions of the quantum Yang–Baxter equation *Commun. Math. Phys.* **148** 309–43
[4] Etingof P, Schedler T and Soloviev A 1999 Set-theoretical solutions to the quantum Yang–Baxter equation *Duke Math. J.* **100** 169–209
[5] Veselov A P 2003 Yang–Baxter maps and integrable dynamics *Phys. Lett.* A **314** 214–21
[6] Veselov A P 2007 Yang–Baxter maps: dynamical point of view *Combinatorial Aspect of Integrable Systems (Kyoto, 2004) (MSJ Mem.* vol 17*)* pp 145–67
    Veselov A P 2007 *Preprint* math/0612814v1 (earlier version)
[7] Etingof P 2003 Geometric crystals and set-theoretical solutions to the quantum Yang–Baxter equation *Commun. Algebra* **31** 1961–73
[8] Adler V E, Bobenko A I and Suris Yu B 2004 Geometry of Yang–Baxter maps: pencils of conics and quadrirational mappings *Commun. Anal. Geom.* **12** 967–1007
[9] Adler V E, Bobenko A I and Suris Yu B 2003 Classification of integrable equations on quad-graphs: the consistency approach *Commun. Math. Phys.* **233** 513–43
[10] Papageorgiou V G, Tongas A G and Veselov A P 2006 Yang–Baxter maps and symmetries of integrable equations on quad-graphs *J. Math. Phys.* **47** 083502 1–16
[11] Adler V E and Yamilov R I 1994 Explicit auto-transformations of integrable chains *J. Phys. A: Math. Gen.* **27** 477–92
[12] Papageorgiou V G, Nijhoff F W and Capel H W 1990 Integrable mappings and nonlinear integrable lattice equations *Phys. Lett.* **147A** 106–14
[13] Nijhoff F W 2002 Lax pair for the Adler (lattice Krichever–Novikov) system *Phys. Lett.* A **297** 49–58
[14] Adler V E and Veselov A P 2004 Cauchy problem for integrable discrete equations on quad-graphs *Acta Appl. Math.* **84** 237–62
[15] Chen H, Lee Y C and Liu C S 1979 Integrability of nonlinear systems by inverse scattering method *Phys. Scr.* **20** 490–2
[16] Kaup D J 1975 Finding eigenvalue problems for solving nonlinear evolution equations *Prog. Theor. Phys.* **54** 72–8
[17] Bobenko A I and Suris Yu B 2002 Integrable systems on quad-graphs *Int. Math. Res. Not.* **11** 573–611
[18] Suris Yu B and Veselov A P 2003 Lax pairs for Yang–Baxter maps *J. Nonlinear Math. Phys.* **10** (Suppl. 2) 223–30
[19] Adler V E 1994 Nonlinear superposition principle for the Jordan NLS equation *Phys. Lett.* A **190** 53–8
[20] Manakov S V 1973 On the theory of two-dimensional stationary self-focusing of electro-magnetic waves *Ž. Èksper. Teoret. Fiz.* **65** 505–16
    Manakov S V 1974 *Sov. Phys.—JEPT* **38** 248–53 (Engl. Transl.)

[21] Ablowitz M J, Prinari B and Trubatch A D 2004 Soliton interactions in the vector NLS equation *Inverse Problems* **20** 1217–37

[22] Tsuchida T 2004 N-soliton collision in the Manakov model *Prog. Theor. Phys.* **111** 151–82

[23] Ablowitz M J, Prinari B and Trubatch A D 2006 Discrete vector solitons: composite solitons, Yang–Baxter maps and computation *Stud. Appl. Math.* **116** 97–133

[24] Reshetikhin N and Veselov A 2005 Poisson Lie groups and Hamiltonian theory of the Yang–Baxter maps *Preprint* math.QA/0512328v2